JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **222,** 460–483 (1998)
ARTICLE NO. AY985934

A Regularity Theorem for Certain $2n$th-Order Nonlinear Differential Operator with a Parameter

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Received July 5, 1995

A regularity theorem for certain $2n$th-order nonlinear differential operator B_λ with the parameter λ is presented. From it, we obtain $(n+3)$ homeomorphism classes regarding the operator B_λ in the nonlinear or linear case. This offers a useful and convenient method for illustrating the two-direction stability properties of some objects in their moving processes. © 1998 Academic Press

Key Words: homeomorphism; linear isomorphism; interpolating inequality.

1. INTRODUCTION

Consider the following boundary value problem of a fourth-order symmetric differential equation arising from research on the stability properties of some flying vehicles (such as rockets, missiles, airplanes, etc.) [1–3]:

$$
\begin{gathered}
y^{(4)}(x) - N(x)y'(x) + \left(\lambda^2 + \frac{\lambda}{V_0} N(x)\right) y(x) = f(x), \qquad x \in [0,1], \\
y(x) \in C^4[0,1], \qquad f(x) \in C[0,1], \qquad \lambda \in C, \qquad (1.1) \\
y\,|_{x=0,1} = y'\,|_{x=0,1} = 0 \quad \left(\text{or } y''\,|_{x=0,1} = y^{(3)}\,|_{x=0,1} = 0\right).
\end{gathered}
$$

Here C is the complex field.

Two comments are in order.

(i) When $f(x) = 0$, the eigenvalues of the solution (1.1) about the parameter λ are the corresponding elastic vibration points of these vehicles. Furthermore, the dimension of the corresponding eigensubspace describes the amplitude at this vibration point of the vehicle.

0022-247X/98 $25.00

(ii) The continuous dependence in both directions of the functions $y(x)$ and $f(x)$ (dependent on the parameter λ in (1.1)) describes the two-way stability of the vehicles of the kinds in their moving processes.

Therefore, it is very important to study the problems relevant to the above-mentioned comments (i) and (ii) for the vibration control of these kinds of vehicles and to understand the ranges of their two-way stability while flying. Issue (i) is discussed, e.g., in [4] and [5]. In this paper, we are interested in the problems related to (ii).

It is known that:

(a) The parameter value λ which makes the functions $y(x)$ and $f(x)$ continuously dependent on each other is precisely the regularity point of the fourth-order differential operator

$$A_\lambda \colon K_{(i,j)} \subset C^4[0,1] \to C[0,1], \tag{1.2}$$

where $A_\lambda = d^4/dx^4 + \sum_{i=1}^{3} P_i(x)(d^i/dx^i) + (\lambda^2 + \lambda N(x))I$, I is the identity operator, and

$$K_{(i,j)} = \left\{ y(x) \in C^4[0,1] \middle| y^{(i)}_{x=0,1} = y^{(j)}_{x=0,1} = 0 \right\}, \qquad (i,j) = (0,1) \text{ or } (2,3).$$

(b) For more appropriate description of the flight properties of these flying objects, we should add the nonlinear disturbance on the left side of Eq. (1.1).

This is equivalent to considering the regularity point with respect to the parameter λ of the fourth-order nonlinear differential operator

$$A_\lambda + B \colon K_{(i,j)} \to C[0,1], \tag{1.3}$$

where $B\colon C^4[0,1] \to C[0,1]$ is a nonlinear differential operator whose order is at most 4.

Since the regularity of the operators depends on the topology in the spaces of the domain and range, the following two important problems arise naturally:

Problem I. For the regularities of the operators in (1.2) and (1.3), how can we define the suitable norms $\|\cdot\|_1$ and $\|\cdot\|_2$ for $C^4[0,1]$ and $C[0,1]$, respectively? How can we describe their regularities?

Problem II. For the space pair $(C^4[0,1], C[0,1])$, do there exist sufficiently many norm dualities $\|\cdot\|_1$, $\|\cdot\|_2$ so that, when we consider the regularities of the operators A_λ and $A_\lambda + B$, we can select the appropriate regularity class describing the two-way smooth properties of these kinds of flying vehicles in different concrete cases?

Since A_λ is nonsymmetric and $A_\lambda + B$ is nonlinear, the self-adjoint operator techniques suitable to higher-order symmetric differential operators, Sturm–Liouville theory and Titchmarch's function- theoretic methods [6–8], cannot be applied to the study of the operators A_λ and $A_\lambda + B$.

For Problems I and II, some preliminary studies have been done [9–12]. In the case $(\|\cdot\|_1, \|\cdot\|_2) = (\|\cdot\|_{H^4(0,1)}, \|\cdot\|_{L^2(0,1)})$ ref. 9 contains results about the regularity of the operator A_λ: $(K_{(i,j)}, \|\cdot\|_1) \to (A_\lambda K_{(i,j)}, \|\cdot\|_2)$ in (1.2) for real parameter λ, where $(i,j) = (0,1), (0,2), (1,3)$, and $(2,3)$. References 10 and 11 are devoted to the study of the regularities of the operators

$$A_\lambda + B\colon \left(K_{(i,j)}, \|\cdot\|_1\right) \to \left((A_\lambda + B)K_{(i,j)}, \|\cdot\|_2\right)$$

for the real parameters in case $(\|\cdot\|_1, \|\cdot\|_2)$ is $(\|\cdot\|_{H^2(0,1)}, \|\cdot\|_{H^{-2}(0,1)})$ or $(\|\cdot\|_{H^4(0,1)}, \|\cdot\|_{H^{-4}(0,1)})$, respectively. (For instance, we can take B as

$$By = \sum_{i=1}^{2} g_i(x) \frac{y^{(i)}}{1 + \left(y^{(i)}\right)^2} + \sin(y'' + y'), \text{ or}$$

$$By = \sum_{i=0}^{3} \left[b_i \sin\left(y^{(4-i)}\right) + \frac{c_i\left(y^{(4-i)}\right)^2}{1 + \left(y^{(4-i)}\right)^2} \right] + \sum_{i=1}^{3} Q_i(x) \frac{y^{(i)}}{1 + \left(y^{(i)}\right)^2}$$
$$+ \sin\left(\sum_{i=1}^{3} R_i(x) y^{(i)} \right),$$

$y \in C^4[0,1]$ (where $|b_0| + |c_0| < 1$).) Reference 12 is on the regularity of the $4n$th-order differential operator with respect to the parameter λ,

$$A + \lambda I\colon K_{(i_1,\ldots,i_{2n})} \subset \left(C^{4n}[0,1], \|\cdot\|_{H^{4n}(0,1)}\right) \to \left(C[0,1], \|\cdot\|_{H^{-4n}(0,1)}\right),$$

where $A = d^{4n}/dx^{4n} + \sum_{i=1}^{4n-1} P_i(x) d^i/dx^i + B$; B is a nonlinear differential operator with differential order at most $4n - 1$. (For instance, we can take

$$By = \sum_{i=1}^{4n-1} Q_i(x) \frac{y^{(i)}}{1 + \left(y^{(i)}\right)^2} + \sin\left(\sum_{i=1}^{4n-1} R_i(x) y^{(i)} \right), \qquad y \in C^{4n}[0,1],$$

and $K_{(i_1,\ldots,i_{2n})} = \{y(x) \in C^{4n}[0,1] \mid y^{(i_k)}_{x=0,1} = 0,\ k = 1,\ldots,2n,\ 0 \le i_1 < i_2 < \cdots < i_{2n} \le 4n\}$.)

Nevertheless, the two problems are far from being completely solved. In this article, for the $2n$th-order linear nonsymmetric and nonlinear differential operator B_λ with the parameter $d_0\lambda^2 + \lambda N(x)$, we consider the corresponding regularity problem that contains Problems I and II as special cases. Roughly speaking, in this article, in case B_λ is linear nonsymmetric or nonlinear, we study the regularity of Problems I and II by considering the homeomorphism set of the following operators:

$$\begin{aligned} B_\lambda\colon (K, \|\cdot\|_{n+k}) &\to \left(B_\lambda K, \|\cdot\|_{-(n+k)}\right) \qquad (0 \le k \le n), \\ B_\lambda\colon (K, \|\cdot\|_{2n}) &\to \left(B_\lambda K, \|\cdot\|\right), \\ B_\lambda\colon (K, \|\cdot\|_{C^{2n}}) &\to \left(B_\lambda K, \|\cdot\|_c\right). \end{aligned} \tag{1.4}$$

The core of the study is functional structure and the method used is the technique in empirical estimate and functional analysis. We get $(n+2)$ homeomorphism classes in case B_λ is real and nonlinear, and $(n+3)$ homeomorphism classes in case B_λ is linear and nonsymmetric. In particular, we also get the linear homeomorphism class of the last operator in (1.4). This result is most useful.

2. DEFINITIONS AND LEMMAS

Definition. Let $B_\lambda = A + N_\lambda I$ be a $2n$th-order operator from $C^{2n}[0,1]$ to $C[0,1]$, where:

(a) $A\colon C^{2n}[0,1] \to C[0,1]$ is a linear or nonlinear operator and $I\colon C^{2n}[0,1] \to C[0,1]$ is the identity operator.

(b) $N_\lambda \triangleq N_\lambda(x) = d_0\lambda^2 + \lambda N(x)$, with d_0 a real constant, $\lambda \in C$, and $N(x) \in C[0,1]$ a real function.

For $m = 0, 1, \ldots, 2n$, the space $C^{2n}[0,1]$ can be equipped with the norms $\|\cdot\|_{C^m}$ and $\|\cdot\|_m$ (Sobolev norm), where

$$\|y\|_{C^m} = \sum_{i=0}^{m} \|y^{(i)}\|_c, \qquad \|y\|_m = \left(\sum_{i=0}^{m} \|y^{(i)}\|^2\right)^{1/2}, \qquad y \in C^{2n}[0,1].$$

$(\cdot,\cdot)$ and $\|\cdot\|$ are the scalar product and norm on $L^2(0,1)$, respectively; $\|\cdot\|_c$ is the maximum norm on $C[0,1]$.

We introduce the following notation:

K is a subspace on $C^{2n}[0,1]$.

$\overline{K}^{\|\cdot\|_m}$ and $\overline{K}^{\|\cdot\|_{C^m}}$ are the closures of K in the norms $\|\cdot\|_m$ and $\|\cdot\|_{C^m}$, respectively.

$((\overline{K}^{\|\cdot\|_m})^*, \|\cdot\|_{-m})$ is the dual space of $(\overline{K}^{\|\cdot\|_m}, \|\cdot\|_m)$, where $\|\cdot\|_{-m}$ is the dual norm of $\|\cdot\|_m$.

$\overline{B_\lambda K}^{\|\cdot\|}$, $\overline{B_\lambda K}^{\|\cdot\|_c}$, and $\overline{B_\lambda K}^{\|\cdot\|_{-m}}$ are the closures of $B_\lambda K$ in the norms $\|\cdot\|$, $\|\cdot\|_c$, and $\|\cdot\|_{-m}$, respectively.

$\langle\cdot,\cdot\rangle$ is a functional (that is a "duality pairing").

We will consider the regularity problem for the following operators arising from the operator B_λ mentioned above,

$$(A_k) \qquad B_\lambda\colon (K, \|\cdot\|_{n+k}) \to \left(B_\lambda K, \|\cdot\|_{-(n+k)}\right),$$

$$(B_k) \qquad \tilde{B}_\lambda\colon \left(\overline{K}^{\|\cdot\|_{n+k}}, \|\cdot\|_{n+k}\right) \to \left(\overline{B_\lambda K}^{\|\cdot\|_{-(n+k)}}, \|\cdot\|_{-(n+k)}\right); k = 0, 1, \ldots, n;$$

$$(C_n) \qquad B_\lambda\colon (K, \|\cdot\|_{2n}) \to \left(B_\lambda K, \|\cdot\|\right),$$

$$(D_n) \qquad \overline{B}_\lambda^{(n)}\colon \left(\overline{K}^{\|\cdot\|_{2n}}, \|\cdot\|_{2n}\right) \to \left(\overline{B_\lambda K}^{\|\cdot\|}, \|\cdot\|\right);$$

$$(E_n) \qquad B_\lambda\colon (K, \|\cdot\|_{C^{2n}}) \to \left(B_\lambda K, \|\cdot\|_C\right),$$

$$(F_n) \qquad \hat{B}_\lambda^{(n)}\colon \left(\overline{K}^{\|\cdot\|_{C^{2n}}}, \|\cdot\|_{C^{2n}}\right) \to \left(\overline{B_\lambda K}^{\|\cdot\|_C}, \|\cdot\|_c\right),$$

where $\tilde{B}_\lambda^{(k)}$, $\overline{B}_\lambda^{(n)}$, and $\hat{B}_\lambda^{(n)}$ are the minimal continuous extensions of the operators in (A_k), (C_n), and (E_n), respectively when B_λ is continuous. For convenience, we denote the above-mentioned operators as operator $(A_k), \ldots,$ operator (F_n), respectively.

For the proof of the main theorem, we need the following:

LEMMA 2.1 [13]. *There exists a constant $a > 0$ such that*

$$\|y^{(j)}\|^2 \le a\left(\varepsilon\|y^{(m)}\|^2 + \varepsilon^{-(m-1)}\|y\|^2\right), \qquad j = 1, 2, \ldots, m-1,$$

for $m = 2, \ldots, 2n$, $y(x) \in C^m[0,1]$, and any $\varepsilon \in (0,1]$.

3. MAIN RESULT

To state the main theorem succinctly, let us introduce some more definitions and notation.

DEFINITION. Let $B_\lambda = A + N_\lambda I$ and let K be defined as in Section 2.

(I) $\forall k \in \{0, 1, \ldots, n\}$, we say:

(1) condition (R_k) holds if $N(x) \in C^k[0,1]$ and

$$\left(N_\lambda y, z^{(2k)}\right) = (-1)^k\left((N_\lambda y)^{(k)}, z^{(k)}\right), \qquad y, z \in K;$$

(2) condition (a_k) holds, if $\forall y \in K$,

$$(-1)^{n-k}\operatorname{Re}\left(Ay, y^{(2k)} + (-1)^k b_k y\right) \ge \varepsilon_0\|y^{(n+k)}\|^2 - \sum_{i=0}^{n+k-1} M_i\|y^{(i)}\|^2;$$

(3) condition $(a_k)'$ holds, if $\forall y, z \in K$,

$$(-1)^{n-k}\operatorname{Re}\Big(Ay - Az, y^{(2k)} - z^{(2k)} + (-1)^k b_k(y-z)\Big)$$
$$\geq \varepsilon_0 \|y^{(n+k)} - z^{(n+k)}\|^2 - \sum_{i=0}^{n+k-1} M_i \|y^{(i)} - z^{(i)}\|^2;$$

(4) condition (b_k) holds, if $\forall y, z, u \in K$,

$$\Big|\Big(Ay - Az, u^{(2k)} + (-1)^k b_k u\Big)\Big| \leq M\|y-z\|_{n+k}\|u\|_{n+k},$$

where b_k, ε_0, M, M_i are constants: $b_0 = 0$, $b_k = 1$ $(k = 1, 2, \ldots, n)$, $\varepsilon_0 > 0$, $M \geq 0$, $M_i \geq 0$ $(i = 0, 1, \ldots, n+k-1)$.

(II) Letting $k = n$, we say that

(1) condition (d_n) holds if $\|Ay - Az\| \leq M'\|y - z\|_{2n}$, $y, z \in K$;

(2) condition (e_n) holds if $\|Ay - c_0 y^{(2n)}\|_C \leq M''\|_{C^{2n-1}}$, $y \in K$, where M' and M'' are nonnegative constants; c_0 is a nonzero constant.

(III) Assume conditions (R_k), (a_k), and $(a_k)'$ hold, $k \in \{0, \ldots, n\}$. We define the following set of complex numbers for a given k and subspace K:

$$C_k(K) = \Bigg\{\lambda \in C \,\Bigg|\, \min_{x\in[0,1]} (-1)^n \operatorname{Re} N_\lambda - \frac{1}{2}(a+1)b_k \sum_{j=1}^{k-1}\binom{k}{j}\|N_\lambda^{(k-j)}\|_c - \frac{1}{2}b_k\|N_\lambda^{(k)}\|_c \geq M_0 + \frac{\varepsilon_0}{2} + ab\left(\frac{2ab}{\varepsilon_0} + 1\right)^{n+k-1}\Bigg\},$$

where $b = \sum_{i=1}^{n+k-1}(M_i + \varepsilon_0/2)$, and a, b_k, M_i $(i = 0, 1, \ldots, n+k-1)$, ε_0 are the constants in Lemma 2.1 and condition (a_k).

We have the following main result on the homeomorphisms and linear homeomorphisms of the operators $(A_k), \ldots, (F_n)$ introduced in Section 2.

MAIN THEOREM. *Let $B_\lambda = A + N_\lambda$. Let K be defined as in Section* 2.

(A) $\forall k \in \{0, 1, \ldots, n\}$, *if conditions* (R_k), $(a_k)'$, *and* (b_k) *hold, then*:

(i) $\forall \lambda \in C_k(K)$, *the operators* (A_k) *and* (B_k) *are homeomorphisms*;

(ii) *if A is a linear operator, then for the same λ as in* (i), *the above-mentioned operators are linear homeomorphisms and*

$$\overline{B_\lambda K}^{\|\cdot\|_{-(n+k)}} = \left(\overline{K}^{\|\cdot\|_{n+k}}\right)^*; \qquad (*)$$

(iii) *if* $C^{2n}[0,1]$ *and* A *are a real space and a real operator, respectively, and condition* (a_k) *holds, then* $\forall\lambda \in C_k(K) \cap R$, *the operators* (A_k) *and* (B_k) *are real homeomorphisms and* $(*)$ *holds.*

When $k = n$ *and condition* (R_n) *and* $(a_n)'$ *hold.*

(B) *If condition* (d_n) *holds, then* $\forall\lambda \in C_n(K)$ *the operators* (C_n) *and* (D_n) *are all homeomorphisms. Furthermore, if* A *is linear, these operators are linear for the same* λ.

(C) *If* A *is a linear operator and condition* (e_n) *holds, then* $\forall\lambda \in C_n(K)$ *the operators* (E_n) *and* (F_n) *are linear homeomorphisms.*

4. PROOF OF THE THEOREM

The theorem will be proved by using three lemmas on operator homeomorphisms and one lemma on estimate.

Suppose K is a subspace on $C^{2n}[0,1]$ and the operator $F\colon K \to C[0,1]$ satisfies the following conditions: $\forall k \in \{0,1,\dots,n\}$ there exist the real constants $\varepsilon > 0$ and δ_i $(i = 0,\dots,2k,\ \delta_{2k} \neq 0)$ which depend on K and k only, such that

$$\operatorname{Re}\left(Fy - Fz,\ \sum_{i=0}^{2k} \delta_i\big(y^{(i)} - z^{(i)}\big)\right) \geq \varepsilon\|y - z\|_{n+k}^2, \qquad y, z \in K. \quad (*)_k$$

LEMMA 4.1. *Suppose* $k \in \{0,1,\dots,n\}$ *and there exists a constant* $M \geq 0$ *such that*

(a) $|(Fy - Fz, \sum_{i=0}^{2k} \delta_i u^{(i)})| \leq M\|y - z\|_{n+k}\|u\|_{n+k}$, $y, z, u \in K$. *Then we have*

(i) *The following two operators are homeomorphisms,*

$$F\colon \big(K, \|\cdot\|_{n+k}\big) \to \big(FK, \|\cdot\|_{-(n+k)}\big)$$

$$\tilde{F}\colon \big(\overline{K}^{\|\cdot\|_{n+k}}, \|\cdot\|_{n+k}\big) \to \big(\overline{FK}^{\|\cdot\|_{-(n+k)}}, \|\cdot\|_{-(n+k)}\big),$$

where $\overline{FK}^{\|\cdot\|_{-(n+k)}}$ *is the closure with respect to the norm* $\|\cdot\|_{-(n+k)}$ *of* FK; $\tilde{F}$ *is the minimal continuous extension.*

(ii) *If* F *is a linear operator, then both of the operators in* (i) *are linear homeomorphisms. Furthermore,*

$$\overline{FK}^{\|\cdot\|_{-(n+k)}} = (\overline{K}^{\|\cdot\|_{n+k}})^*. \tag{4.1}$$

(iii) *If* K *and* F *are a real space and a real operator, respectively, and satisfy* $F0 = 0$.

(b) *Alternatively, if* $(Fy, \sum_{i=0}^{2k} \delta_i y^{(i)}) \geq \varepsilon' \|y\|_{n+k}^2$, $y \in K$, *for some constant* $\varepsilon' > 0$, *then both operators in* (i) *are real homeomorphisms and* (4.1) *holds*.

Proof. $\forall f(x) \in C[0,1]$, define the linear functional on $\overline{K}^{\|\cdot\|_{n+k}}$ as

$$\langle f, y \rangle = \left(f, \sum_{i=0}^{2k} \delta_i \bar{y}^{(i)} \right), \qquad y \in \overline{K}^{\|\cdot\|_{n+k}}. \tag{4.2}$$

Then by conditions $(*)_k$, Lemma 4.1(a), and Schwarz inequality, with $2k \leq n+k$, we have

$$\tilde{F} \text{ is a continuous operator, } \tilde{F}\overline{K}^{\|\cdot\|_{n+k}} \subseteq (\overline{K}^{\|\cdot\|_{n+k}})^*,$$

$$\begin{aligned} \varepsilon \|y - z\|_{n+k} &\leq \|\tilde{F}y - \tilde{F}z\|_{-(n+k)} \\ &\leq M\|y - z\|_{n+k}, \qquad y, z \in \overline{K}^{\|\cdot\|_{n+k}}, \end{aligned} \tag{4.3}$$

$$\tilde{F}\overline{K}^{\|\cdot\|_{n+k}} = \overline{FK}^{\|\cdot\|_{-(n+k)}}.$$

So $\tilde{F}$ is homeomorphism and (i) holds.

As for (ii) and (iii), we only prove (iii); the proof for (ii) is similar [14].

Under the condition (iii), $\tilde{F}$ is a real homeomorphism, so it is sufficient to prove (4.1). From (4.3)

$$\overline{FK}^{\|\cdot\|_{-(n+k)}} = \tilde{F}\overline{K}^{\|\cdot\|_{n+k}} \subseteq (\overline{K}^{\|\cdot\|_{n+k}})^*.$$

So we need only show that the following real continuous operator is a surjection:

$$\tilde{F}\colon \left(\overline{K}^{\|\cdot\|_{n+k}}, \|\cdot\|_{n+k}\right) \to \left((\overline{K}^{\|\cdot\|_{n+k}})^*, \|\cdot\|_{-(n+k)}\right). \tag{4.4}$$

Let $\varepsilon_0 = \min\{\varepsilon, \varepsilon'\}$. From $(*)_k$, Lemma 4.1(b), and (4.2), we have

$$\left.\begin{aligned} &\langle \tilde{F}y - \tilde{F}z, y - z \rangle \geq \varepsilon_0 \|y - z\|_{n+k}^2 && \text{(monotonicity)}, \\ &\langle \tilde{F}y, y \rangle \geq \varepsilon_0 \|y\|_{n+k}^2 && \text{(coerciveness)}, \end{aligned}\right. \quad y, z \in \overline{K}^{\|\cdot\|_{n+k}}.$$

So the operator $\tilde{F}$ in (4.4) is surjective [15].

LEMMA 4.2. *Assume* $k = n$, *and for some constant* $M' \geq 0$,

(c) $\|Fy - Fz\| \leq M'\|y - z\|_{2n}$, $y, z \in K$. *Then*

(i) *Both of the following operators are homeomorphisms,*

$$F\colon (K, \|\cdot\|_{2n}) \to (FK, \|\cdot\|),$$

$$\bar{F}\colon \left(\overline{K}^{\|\cdot\|_{nn}}, \|\cdot\|_{2n}\right) \to \left(\overline{FK}^{\|\cdot\|}, \|\cdot\|\right),$$

where $\overline{FK}^{\|\cdot\|}$ *is FK's closure in the norm* $\|\cdot\|$ *and* $\overline{F}$ *is F's minimal continuous extension.*

(ii) *If F is a linear operator, then both of the operators in* (i) *are linear homeomorphisms.*

Lemma 4.2 can be easily proved. We notice that $(*)_n$ holds, from the definition of F, in case $k = n$. So

$$\|Fy - Fz\| \geq \left(\sum_{i=0}^{2n} |\delta_i| \right)^{-1} \varepsilon \|y - z\|_{2n}.$$

LEMMA 4.3. *Assume* $k = n$, *and for some nonnegative integer* $\mu < 2n$, *there are real constants* c_i $(i = 0, 1, \dots, \mu)$ $(c_0 \neq 0)$ *and* $M'' \geq 0$, *such that*

(d) $\|Fy - \sum_{i=0}^{\mu} c_i y^{(2n-i)}\|_c \leq M'' \|y\|_{c^{2n-\mu-1}}$. *Then the following two operators are linear homeomorphisms,*

$$F\colon (K, \|\cdot\|_{c^{2n}}) \to (FK, \|\cdot\|_c),$$
$$\hat{F}\colon \left(\overline{K}^{\|\cdot\|_{c^{2n}}}, \|\cdot\|_{c^{2n}}\right) \to \left(\overline{FK}^{\|\cdot\|_c}, \|\cdot\|_c\right),$$

where $\overline{FK}^{\|\cdot\|_c}$ *is the closure of FK in the norm* $\|\cdot\|_c$, *and* $\hat{F}$ *is the minimal continuous extension of* F.

Proof. Noticing $k = n$, by linearity of F, conditions $(*)_n$, and Lemma 4.3(d), we have:

$$\hat{F}\colon \left(\overline{K}^{\|\cdot\|_{c^{2n}}}, \|\cdot\|_{c^{2n}}\right) \to \left(\hat{F}\overline{K}^{\|\cdot\|_{c^{2n}}}, \|\cdot\|_c\right) \text{ is a linear continuous operator,} \tag{4.5a}$$

$$\hat{F}\overline{K}^{\|\cdot\|_{C^{2n}}} \subset \overline{FK}^{\|\cdot\|_C} \subset C[0,1], \tag{4.5b}$$

$$\|\hat{F}y\|_c \geq \left(\sum_{i=0}^{2n} |\delta_i| \right)^{-1} \varepsilon \|y\|_{2n}. \tag{4.5c}$$

From the inverse mapping theorem, we only need to prove

$$\overline{FK}^{\|\cdot\|_c} \subset \hat{F}\overline{K}^{\|\cdot\|_{c^{2n}}}. \tag{4.6}$$

We prove this result in two steps.

Step 1. If $\{y_m\} \subset K$, such that $\|Fy_m\|_c \to 0$ $(m \to \infty)$, then

$$\|y_m\|_{c^{2n}} \to 0 \qquad (m \to \infty). \tag{4.7}$$

This implies that if $\{y_m\} \subset K$ and $\{Fy_m\}$ is a Cauchy sequence in the norm $\|\cdot\|_c$, then $\{y_m\}$ is a Cauchy sequence in the norm $\|\cdot\|_{c^{2n}}$. From Lemma 4.3(d), we have $c_0 \neq 0$, and $\forall y \in K$,

$$|c_0| \|y_m^{(2n)}\|_c \leq \|Fy_m\|_c + \sum_{i=1}^{\mu} |c_i| \, \|y_m^{(2n-i)}\|_c + M'' \|y_m\|_{c^{2n-\mu-1}}.$$

(If $\mu = 0$, the item $\sum_{i=1}^{\mu} |c_i| \, \|y_m^{(2n-i)}\|_c$ disappears.)

So we only need to prove

$$\|y_m^{(i)}\|_c \to 0 \qquad (m \to \infty), \ i = 0, \dots, 2n-1.$$

For a given $i \in \{0, 1, \dots, 2n-1\}$, by $(4.5)_c$ and Schwarz inequality, we have

$$|y_m^{(i)}(x) - y_m^{(i)}(0)| \leq \|y_m\|_{2n} \to 0$$

$$(m \to \infty) \text{ holds uniformly for all } x \in [0,1], \tag{4.8a}$$

$$\|y_m^{(i)}\|_c \leq |y_m^{(i)}(0)| + \|y_m\|_{2n}, \qquad m = 1, 2, \dots . \tag{4.8b}$$

Hence, we only need to prove $y_m^{(i)}(0) \to 0$ $(m \to \infty)$. We argue by contradiction. Suppose, without lost of generality, there exists a constant $\delta > 0$, such that

$$|y_m^{(i)}(0)| \geq \delta > 0, \qquad m = 1, 2, \dots .$$

Combined with (4.8a), there exists a natural number N such that

$$\|y_m^{(i)}\| \geq \frac{\delta}{2} > 0, \qquad m \geq N.$$

This contradicts the fact $\|y_m\|_{2n} \to 0$ $(m \to 0)$. So (4.7) holds.

Step 2. Now we prove (4.6). Assume $f_0 \in \overline{FK}^{\|\cdot\|_c}$. Then there exists $\{y_m\} \subset K$, such that

$$\|Fy_m - f_0\|_c \to 0 \qquad (m \to \infty).$$

By (4.7), $\{y_m\}$ is a Cauchy sequence in the norm $\|\cdot\|_{c^{2n}}$. So there exists $y_0 \in \overline{K}^{\|\cdot\|_{c^{2n}}}$, such that

$$\|y_m - y_0\|_{c^{2n}} \to 0 \qquad (m \to \infty).$$

Then $\|Fy_m - \hat{F}y_0\|_c \to 0$ $(m \to \infty)$ and

$$f_0 = \hat{F}y_0 \in \hat{F}\overline{K}^{\|\cdot\|_{c^{2n}}}.$$

LEMMA 4.4. *Assume $k \in \{0, 1, \dots, n\}$, condition (R_k) holds, and $\lambda \in C_k(K)$. Then if condition (a_k) or $(a_k)'$ holds, then for $y, z \in K$, we have*

$$(-1)^{n-k} \operatorname{Re}\Big(B_\lambda y, y^{(2k)} + (-1)^k b_k y\Big) \geq \frac{\varepsilon_0}{2}\|y\|_{n+k}^2, \tag{4.9a}$$

$$\begin{aligned}(-1)^{n-k} &\operatorname{Re}\Big(B_\lambda y - B_\lambda z, y^{(2k)} - z^{(2k)} + (-1)^k b_k (y - z)\Big) \\ &\geq \frac{\varepsilon_0}{2}\|y - z\|_{n+k}^2.\end{aligned} \tag{4.9b}$$

Proof. It is enough to prove (4.9a) under the assumptions (R_k) and (a_k). The proof for (4.9b) is similar.

By conditions (R_k) and (a_k),

$$(N_\lambda y)^{(k)} = N_\lambda y^{(k)} + b_k \sum_{i=0}^{k-1} \binom{k}{i} N_\lambda^{(k-i)} y^{(i)}, \qquad y \in K.$$

(The second item on the right-hand side of the equation disappears when $k = 0$.) It follows that

$$\begin{aligned}
(-1)^{n-k} &\operatorname{Re}\Big(B_\lambda y, y^{(k)} + (-1)^k b_k y\Big) \\
&= (-1)^{n-k} \operatorname{Re}\Big(Ay, y^{(2k)} + (-1)^k b_k y\Big) \\
&\quad + (-1)^{n-k} \operatorname{Re}\Big(N_\lambda y, y^{(2k)} + (-1)^k b_k y\Big) \\
&\geq \frac{\varepsilon_0}{2}\Big(\|y\|_{n+k}^2 + \|y^{(n+k)}\|^2\Big) \\
&\quad - \sum_{i=1}^{n+k-1}\left(M_i + \frac{\varepsilon_0}{2}\right)\|y^{(i)}\|^2 - \frac{b_k}{2}\sum_{i=1}^{k-1}\binom{k}{i}\|N_\lambda^{(k-i)}\|_c \|y^{(i)}\|^2 \\
&\quad + \int_0^1 \left[(-1)^n \operatorname{Re} N_\lambda - \frac{b_k}{2}\sum_{i=0}^{k-1}\binom{k}{i}\|N_\lambda^{(k-i)}\|_c\right]|y^{(k)}|^2\, dx \\
&\quad + \int_0^1 \left[(-1)^n b_k \operatorname{Re} N_\lambda - \frac{b_k}{2}\|N_\lambda^{(k)}\|_c - M_0 - \frac{\varepsilon_0}{2}\right]|y|^2\, dx.
\end{aligned} \tag{4.10}$$

For $\|y^{(i)}\|^2$ $(i = 1, \dots, n + k - 1\}$, $m = n + k$, $\varepsilon = (2ab/\varepsilon_0 + 1)^{-1}$, and $\|y^{(i)}\|^2$ $(i = 1, \dots, k - 1)$, $m = k$ and $\varepsilon = 1$, by Lemma 2.1, we obtain

$$\|y^{(i)}\|^2 \le a\left[\left(\frac{2ab}{\varepsilon_0} + 1\right)^{-1} \|y^{(n+k)}\|^2 + \left(\frac{2ab}{\varepsilon_0} + 1\right)^{n+k-1} \|y\|^2\right] \qquad (1 \le i \le n + k - 1), \tag{4.11a}$$

$$\|y^{(i)}\|^2 \le a\left[\|y^{(k)}\|^2 + \|y\|^2\right], \qquad 1 \le i \le k - 1 \tag{4.11b}$$

$\forall y \in C^{2n}[0, 1]$, where $b = \sum_{i=1}^{n+k-1}(M_i + \varepsilon_0/2)$.

From (4.10) and (4.11), $\forall\lambda \in C_k(K)$, $y \in K$, we have

$$\begin{aligned}
&(-1)^{n-k} \operatorname{Re}\left(B_\lambda y, y^{(2k)} + (-1)^k b_k y\right) \\
&\quad \ge \frac{\varepsilon_0}{2}\|y\|_{n+k}^2 + \int_0^1 \Bigg[(-1)^n \operatorname{Re} N_\lambda - \frac{1}{2}(a + 1) b_k \\
&\qquad\qquad \times \sum_{i=1}^{k-1} \binom{k}{i} \|N_\lambda^{(k-i)}\|_c - \frac{1}{2} b_k \|N_\lambda^{(k)}\|_c\Bigg] |y^{(k)}|^2 \, dx \\
&\qquad + \int_0^1 \Bigg[(-1)^n b_k \operatorname{Re} N_\lambda - \frac{1}{2} a b_k \sum_{i=1}^{k-1} \binom{k}{i} \|N_\lambda^{(k-i)}\|_c \\
&\qquad\qquad - \frac{b_k}{2}\|N_\lambda^{(k)}\|_c - M_0 - \frac{\varepsilon_0}{2} - ab\left(\frac{2ab}{\varepsilon_0} + 1\right)^{n+k-1}\Bigg] |y|^2 \, dx \\
&\quad \ge \frac{\varepsilon_0}{2}\|y\|_{n+k}^2.
\end{aligned}$$

Thus, Lemma 4.4 is proved.

Proof of the Main Theorem. Let $k = 0, 1, \dots, n$, and

$$F = B_\lambda, \qquad \delta_0 = (-1)^n b_k, \qquad \delta_i = 0 \qquad (i = 1, \dots, 2k - 1),$$
$$\delta_{2k} = (-1)^{n-k}.$$

(I) The condition (b_k), (d_n), and (e_n) in the theorem imply that for each $\lambda \in C$,

$$\left|\left(Fy - Fz, \sum_{i=0}^{2k} \delta_i u^{(i)}\right)\right| \le \left(2\|N_\lambda\|_c + M\right)\|y - z\|_{n+k}\|u\|_{n+k}, \qquad y, z, u \in K,$$

$$\|Fy - Fz\| \le \left(\|N_\lambda\|_c + M'\right)\|y - z\|_{2n}, \qquad y, z \in K,$$

$$\|Fy - c_0 y^{(2n)}\|_c \le \left(\|N_\lambda\|_c + M''\right)\|y\|_{c^{2n-1}}, \qquad y \in K.$$

Therefore, conditions (a), (c), and (d) in Lemmas 4.1–4.3 hold.

(II) If conditions (R_k) holds, $\forall\lambda \in C_k(K)$, using Lemma 4.4, the conditions $(a_k)'$ and (a_k) in the theorem yield

$$\left.\begin{aligned} &\operatorname{Re}\left(Fy - Fz, \sum_{i=0}^{2k} \delta_i\left(y^{(i)} - z^{(i)}\right)\right) \ge \frac{\varepsilon_0}{2}\|y - z\|_{n+k}^2, \\ &\operatorname{Re}\left(Fy, \sum_{i=0}^{2k} \delta_i y^{(i)}\right) \ge \frac{\varepsilon_0}{2}\|y\|_{n+k}^2, \end{aligned}\right\} \qquad y, z \in K.$$

Therefore, $(*)_k$ about operator F and condition (b) in Lemma 4.1 hold.

From the above argument, we obtain (A), (B), and (C) in the main Theorem by Lemmas 4.1–4.3, respectively.

Remark 4.1. From the Main Theorem and its proof, we know

(a) When dual space $(K, B_\lambda K)$ is equipped with the $(n + 3)$ norm dualities as mentioned in Section 2, then, under some suitable conditions, there exist at least $(n + 2)$ homeomorphisms for the nonlinear operator $B_\lambda\colon K \to B_\lambda K$; there exist $(n + 3)$ homeomorphisms for the linear (non-symmetric) operator $B_\lambda\colon K \to B_\lambda K$.

(b) When $k \in \{0, 1, 2, \dots, n\}$ and conditions (R_k) hold, the set $C_k(K)$ characterizing the range of B_λ's regularity points mainly depends on the conditions (a_k) and $(a_k)'$, and these conditions essentially depend on the coefficient of the operator $B_\lambda = A + N_\lambda I$ (especially, the operator A) itself.

(c) The key to obtain the different regularity classes of the operator B_λ is the functional structure (4.2) in the proof of Lemma 4.1. The $(n + 3)$ norm dualities in (1.4) are just deducted from (4.2). Furthermore, from Lemma 2.1, we also obtain the duality pairings

$$\langle f, y\rangle = \left(f, \sum_{i=0}^{2k} \delta_i \bar{y}^{(i)}\right) \quad \text{and} \quad \langle f, y\rangle = \left(f, \delta_0\bar{y} + \delta_{2k}\bar{y}^{(2k)}\right), \qquad y \in K.$$

These pairings are equivalent for the purposes of obtaining the conditions (a_k) and $(a_k)'$. In addition, the only form of other pairings is $\langle f, y\rangle = (f, \sum_{i=0}^{2k-1} \delta_i \bar{y}^{(i)})$, $f \in C[0,1]$, $y \in K$. However from this kind of duality pairing we cannot obtain the conditions similar to (a_k) and $(a_k)'$ that make B_λ homeomorphic or linear homeomorphic, so in the view of duality pairing, the homeomorphisms (or linear homeomorphisms) relevant to B_λ we can get are at most $(n+2)$ (or $(n+3)$).

Hence, the Main Theorem solves the regularity problem corresponding to the general $2n$th-order linear or nonlinear operator B_λ: $K \to B_\lambda K$ which contains Problems I and II as its special cases.

Remark 4.2. For the regularity point set $C_k(K)$.

(a) Let $d_0 > 0$, $\lambda = \lambda_1 + \lambda_2 i$ in N_λ. Then $(-1)^n \operatorname{Re} N_\lambda = (-1)^n(d_0(\lambda_0^2 - \lambda_2^2) + \lambda_1 N(x))\ldots$. This indicates that the regularity point set of B_λ includes, at least, the interior of a hyperbola on a complex plane. In particular, when n is an even integer, it includes all real numbers on the real line except a finite interval; when n is an odd integer, it includes all pure imaginary numbers on the axis of imaginaries exception to a finite interval.

(b) Take $d_0 = 0$, $N(x) \equiv 1$. Then $N_\lambda \equiv \lambda$, and so

$$C_k(K) = \left\{ \lambda \in C \mid (-1)^n \operatorname{Re} \lambda \geq M_0 + \frac{\varepsilon_0}{2} + ab\left(\frac{2ab}{\varepsilon_0} + 1\right)^{n+k-1} \right\}.$$

Hence, $C_k(K)$ is the left half-plane of the complex plane C when n is odd and the right half-plane when n is even. Obviously, when $n = 2k$ or $2k-1$, the ranges of B_λ's regularity points are largely different.

5. COROLLARIES OF THE MAIN THEOREM AND SOME REMARKS

In this section, we present two corollaries of the Main Theorem on the regularities of the fourth- and $2n$th-order differential operators.

Let

$$K_{(i_1,\ldots,i_n)} = \left\{ y(x) \in C^{2n}[0,1] \,\middle|\, y_{x=0,1}^{(i_k)} = 0, k = 1,\ldots,n, \right.$$
$$\left. 0 \leq i_1 < i_2 < \cdots < i_n \leq 2n \right\}, \qquad n = 2,3,\ldots .$$

(A)

Let $k \in \{0,1,2\}$, $N(x) \in C^k[0,1]$, $f(x;\xi_0,\xi_1,\ldots,\xi_4) \in C([0,1]\times C^5)$, and for any $y,z,u \in C^4[0,1]$,

$$\left|\left(f(x;y,y',\ldots,y^{(4)}),\, y^{(2k)} + (-1)^k b_k y\right)\right| \\ \leq \alpha_k \|y^{(2+k)}\|^2 + c_k \|y\|_{2+k}\|y\|_{1+k}, \tag{5.1a}$$

$$\left|\left(f(x;y,y',\ldots,y^{(4)}) - f(x;z,z',\ldots,z^{(4)}),\, u^{(2k)} + (-1)^k b_k u\right)\right| \\ \leq \alpha_k \|y^{(2+k)} - z^{(2+k)}\|\,\|u^{(2+k)}\| + c_k\|y-z\|_{2+k}\|u\|_{1+k}, \tag{5.1b}$$

where b_k is defined as in Section 3, $\alpha_k \in [0,1)$, and $c_k \geq 0$ are constants depending on f and k only.

Define the operators $A_k^{(1)}$, B_k, A_k: $C^4[0,1] \to C[0,1]$ and $B_\lambda^{(k)}$: $K \subset C^4[0,1] \to C[0,1]$ as

$$A_k^{(1)} = \frac{d^4}{dx^4} + \sum_{i=0}^{3} P_i \frac{d^i}{dx^i}, \qquad A_k = A_k^{(1)} + \beta_k B_k, \qquad B_\lambda^{(k)} = A_k + N_\lambda I,$$

where $\beta_k = 0$ or 1, $N_\lambda = \lambda^2 + \lambda N(x)$, $P_i = P_i(x) \in C[0,1]$ $(i = 0,\ldots,3)$ are all real functions, $B_k y = f(x;y,y',\ldots,y^{(4)})$, and $y \in C^4[0,1]$.

Obviously, $A = A_k$ and $B_\lambda = B_\lambda^{(k)}$ are the operators A and B_λ, which are defined as in Section 3 for the special case $n = 2$. To give the solution to the regularity problem of the operators $B_\lambda^{(k)}$: $K \to B_\lambda^{(k)}K$ for $k = 0,1,2$, we use the following notations:

(i) When $k = 0$, then $P_3 \in C^1[0,1]$ and K is limited to $K_{(0,1)}$, $K_{(0,2)}$, $K_{(1,3)}$, or $K_{(2,3)}$. Furthermore, we assume $P_3(0) = P_3(1) = 0$ when $K = K_{(1,3)}$.

(ii) K is limited to $K_{(0,2)}$ or $K_{(1,3)}$ when $k = 1$.

(iii) When $k = 2$, K is limited to $K_{(0,1)}$, $K_{(0,2)}$, or $K_{(2,3)}$.

LEMMA 5.1. (i) *For all $k = 0,1,2$, (R_k) and (b_k) hold, and the operator A_k satisfies the conditions (a_k) and $(a_k)'$ of the main Theorem $(n = 2)$. We can take the relevant constants as*

$$\varepsilon_0 = \frac{1}{2}(1 - \alpha_k\beta_k),$$

$$M_i \equiv \frac{1}{2}(1-\alpha_k\beta_k) + \frac{1}{2(1-\alpha_k\beta_k)} \\ \times\left[b_k + \beta_k c_k + (1+b_k)\sum_{i=0}^{3}\|P_i\|_c + (1-b_k)\|P_3'\|_c\right]^2, \\ i = 0,\ldots,1+k. \tag{5.2}$$

(ii) *When* $k = 2$,

(a) *If f satisfies*

$$\left|f(x; \xi_0, \dots, \xi_4) - f(x; \xi_0', \dots, \xi_4')\right| \le M'(|\xi_0 - \xi_0'| + \cdots + |\xi_4 - \xi_4'|),$$

then the operator A_2 *satisfies the condition* (d_2) *of the Main Theorem.*

(b) *When* $\beta_2 = 0$, *the operator* $A_2 = A_2^{(1)}$ *satisfies* (e_2) *of the Main Theorem. Furthermore,*

$$\|A_2 y - y^{(4)}\|_c \le \left(\sum_{i=0}^{3} \|P_i\|_c \right) \|y\|_{c^3}, \qquad y \in C^4[0,1].$$

Proof. All conditions except (a_k) and $(a_k)'$ can be verified easily. If we take $u = y - z$ in (5.1b), then we can prove $(a_k)'$ and (a_k) with the similar method. So we only prove the condition (a_k) holds.

By using Schwarz's inequality we can show that for any $k = 0, 1, 2$,

$$\left| (-1)^{2-k} \operatorname{Re}\left(\sum_{i=0}^{3} P_i y^{(i)}, y^{(2k)} + (-1)^k b_k y \right) \right|,$$

$$\le \left[(1 + b_k) \sum_{i=0}^{3} \|P_i\|_c + (1 - b_k) \|P_3'\|_c \right] \|y\|_{2+k} \|y\|_{1+k}, \qquad y \in K.$$

Together with (5.1a), we obtain

$$\begin{aligned}
&(-1)^{2-k} \operatorname{Re}\left(A_k y, y^{(2k)} + (-1)^k b_k y \right) \\
&\quad \ge (-1)^{2-k} \operatorname{Re}\left(A_k^{(1)} y, y_y^{(2k)} + (-1)^k b_k y \right) \\
&\qquad - \beta_k \left| \left(B_k y, y^{(2k)} + (-1)^k b_k y \right) \right| \\
&\quad \ge (1 - \alpha_k \beta_k) \|y^{(2+k)}\|^2 \\
&\qquad - \left[b_k + \beta_k c_k + (1 + b_k) \sum_{i=0}^{3} \|P_i\|_c + (1 - b_k) \|P_3'\|_c \right] \|y\|_{2+k} \|y\|_{1+k} \\
&\quad \ge \frac{1 - \alpha_k \beta_k}{2} \|y^{(2+k)}\|^2 \\
&\qquad - \left\{ \frac{1 - \alpha_k \beta_k}{2} + \frac{1}{2(1 - \alpha_k \beta_k)} \right. \\
&\qquad\quad \left. \times \left[b_k + \beta_k c_k + (1 + b_k) \sum_{i=0}^{3} \|P_i\|_c + (1 - b_k) \|P_3'\|_c \right]^2 \right\} \|y\|_{1+k}^2 .
\end{aligned}$$

Thus the (a_k) holds.

For $k \in \{0,1,2\}$, taking $C_k(K)$ as we did in the Main Theorem according to (R_k), (a_k), and $(a_k)'$, we obtain

COROLLARY 1. (i) *When* $k \in \{0,1,2\}$ *and* $\lambda \in C_k(K)$,

(a) *The operator* $B_\lambda^{(k)}\colon (K, \|\cdot\|_{2+k}) \to (B_\lambda^{(k)}K, \|\cdot\|_{-(2+k)})$ *and its minimal continuous extension* $\tilde{B}_\lambda^{(k)}\colon (\overline{K}^{\|\cdot\|_{2+k}}, \|\cdot\|_{2+k}) \to (\overline{B_\lambda^{(k)}K}^{\|\cdot\|_{-(2+k)}}, \|\cdot\|_{-(2+k)})$ *are all homeomorphisms.*

(b) *When* $\beta_k = 0$, *both operators in* (a) *are linear homeomorphisms and*

$$\overline{B_\lambda^{(k)}K}^{\|\cdot\|_{-(2+k)}} = (\overline{K}^{\|\cdot\|_{2+k}})^*. \tag{$*$}'$$

(c) *If* $C^4[0,1]$ *is real space and the function* $f(x;\xi_0,\xi_1,\ldots,\xi_4)$ *is real, then for each* $\lambda \in C_k(K)\cap R$, *both operators in* (a) *are real homeomorphisms and* $(*)'$ *holds also.*

(ii) *When* $k = 2$ *and* $\lambda \in C_2(K)$,

(a) *If* $f(x;\xi_0,\xi_1,\ldots,\xi_4)$ *satisfies the Lipschitz condition with respect to* $\xi_0,\xi_1,\ldots,\xi_4$, *then* $B_\lambda^{(2)}\colon (K,\|\cdot\|_4) \to (B_\lambda^{(2)}K, \|\cdot\|)$ *and its minimal continuous extension* $\overline{B}_\lambda^{(2)}\colon (\overline{K}^{\|\cdot\|_4}, \|\cdot\|_4) \to (\overline{B_\lambda^{(2)}K}^{\|\cdot\|}, \|\cdot\|)$ *are homeomorphisms, and they are linear homeomorphisms in the case* $\beta_2 = 0$.

(b) *When* $\beta_2 = 0$, *then* $B_\lambda^{(2)}\colon (K, \|\cdot\|_{c^4}) \to (B_\lambda^{(2)}K, \|\cdot\|_c)$ *and its minimal continuous extension* $\hat{B}_\lambda^{(2)}\colon (\overline{K}^{\|\cdot\|_{c^4}}, \|\cdot\|_c) \to (\overline{B_\lambda^{(2)}K}^{\|\cdot\|_c}, \|\cdot\|_c)$ *are all linear homeomorphisms.*

The corollary can be easily proved by applying Lemma 5.1 *and the Main Theorem with* $n = 2$.

Remark 5.1. We can get the results of [9–11] from Corollary 1 by taking $\beta_2 = 0$ in (ii)(a), $k = 0$ in (i)(c), and $k = 2$ in (i)(c), respectively.

Remark 5.2. Corollary 1 shows that the regularities of the different operators $B_\lambda^{(k)}$ and the subspace K relevant to the boundary condition can describe the flight stability of certain flying vehicles in different backgrounds [1–3, 16].

Remark 5.3. We can extend Corollary 1 to the case of $2n$th-order differential operator. Naturally, we should take the subspace K in the form of $K_{(i_1,i_2,\ldots,i_n)}$.

(B)

Let $k \in \{0,1,\ldots,n\}$, $N(x) \in C^k[0,1]$, $A_k^{(1)} = \alpha_k(d^{2n}/dx^{2n}) + \sum_{i=0}^{2n-1} P_i(d^i/dx^i)$, and $A_k^{(2)}\colon C^{2n}[0,1] \to C[0,1]$ such that for any $y, z, u \in$

$C^{2n}[0,1]$,

$$\left|\left(A_k^{(2)}y, y^{(2k)} + (-1)^k b_k y\right)\right| \le \varepsilon_k' \|y^{(n+k)}\|^2 + \sum_{i=0}^{n+k-1} M_i' \|y^{(i)}\|^2, \quad (5.3a)$$

$$\left|\left(A_k^{(2)}y - A_k^{(2)}z, u^{(2k)} + (-1)^k b_k u\right)\right|$$
$$\le \varepsilon_k' \|y^{(n+k)} - z^{(n+k)}\| \, \|u^{(n+k)}\| + \sum_{i=0}^{n+k-1} M_i' \|y^{(i)} - z^{(i)}\| \, \|u^{(i)}\| \quad (5.3b)$$

with the real constants $\alpha_k > 0$, $0 \le \varepsilon_k' < \alpha_k$, and $M_i' \ge 0$ $(i = 0, \dots, n + k - 1)$, and $P_i \in C[0,1]$ $(i = 0, \dots, n + k - 1)$, $P_i \in C^{i-(n+k)}[0,1]$ $(i = n + k, \dots, 2n - 1)$ are all real functions. We define

$$A_k = A_k^{(1)} + \beta_k A_k^{(2)}, \qquad B_\lambda^{(k)} = A_k + N_\lambda I,$$

where $\beta_k = 0$ or 1, $N_\lambda = d_0 \lambda^2 + \lambda N(x)$, $\lambda \in C$.

Consider the regularity of the operator $B_\lambda^{(k)}$: $K \subset C^{2n}[0,1] \to C[0,1]$ in the case $k = 0, 1, \dots, n$. For the sake of simplicity, we only take

$$K = \begin{cases} K_{(0,1,\dots,n-1)} \text{ or } K_{(n,n+1,\dots,2n-1)}, \\ \qquad k = 0, n, \\ K_{(0,1,\dots,k-1,n+k,\dots,2n-1)} \text{ or } K_{(k,k+1,\dots,2k-1,n+k,\dots,2n-1)}, \\ \qquad k = 1, \dots, n-1. \end{cases} \quad (5.4)$$

LEMMA 5.2. (i) *Assume $k \in \{0, 1, 2, \dots, n\}$. Then for the operator $B_\lambda = B_\lambda^{(k)}$: $K \to C[0,1]$, the conditions (R_k), (a_k), $(a_k)'$, and (b_k) in the Main Theorem hold. In particular, the constants in (a_k) and $(a_k)'$ can be taken as $\varepsilon_0 = \frac{1}{2}(\alpha_k - \beta_k \varepsilon_k')$, and*

$$M_\mu = \beta_k M_\mu' + \frac{1}{2(\alpha_k - \beta_k \varepsilon_k')} \times \left[\alpha_k b_k + (1 + b_k)\left(\sum_{i=n+k}^{2n-1} \sum_{j=0}^{i-n-k} \binom{i-n-k}{j} \times \|P_i^{(j)}\|_c + \sum_{i=0}^{n+k-1} \|P_i\|_c\right)\right]^2 + \frac{\alpha_k - \beta_k \varepsilon_k'}{2}, \qquad \mu = 0, \dots, n + k - 1.$$

(ii) *When* $k = n$

(a) *If* $A_n^{(2)}$ *satisfies* (*the Lipschitz condition*)

$$\|A_n^{(2)}y - A_n^{(2)}z\| \le M'\|y - z\|_{2n}, \qquad y, z \in K,$$

then (d_n) *holds*.

(b) *When* $\beta_n = 0$, (e_n) *holds for* $A_n = A_n^{(1)}$. *In particular*,

$$\|A_n y - \alpha_n y^{(2n)}\|_c \le \left(\sum_{i=0}^{2n-1} \|P_i\|\right)\|y\|_{C^{2n-1}}, \qquad y \in K.$$

Proof. We only prove the conditions (a_k) and $(a_k)'$.
From (5.3) and (5.4) we can show that for any $y \in K$,

$$\begin{aligned}
&(-1)^{n-k}\operatorname{Re}\left(A_k y, y^{(2k)} + (-1)^k b_k y\right)\\
&\ge (\alpha_k - \beta_k\varepsilon_k')\|y^{(n+k)}\|^2 - \beta_k\sum_{i=0}^{n+k-1} M_i'\|y^{(i)}\|^2\\
&\quad - \Bigg\{\alpha_k b_k + (1 + b_k)\\
&\qquad \times\left[\sum_{i=n+k}^{2n-1}\sum_{j=0}^{i-n-k}\binom{i-n-k}{j}\|P_i^{(j)}\|_c + \sum_{i=0}^{n+k-1}\|P_i\|_c\right]\Bigg\}\\
&\quad \times\|y\|_{n+k}\|y\|_{n+k-1}\\
&\ge \frac{1}{2}(\alpha_k - \beta_k\varepsilon_k')\|y^{(n+k)}\|^2\\
&\quad - \sum_{\mu=0}^{n+k-1}\Bigg\{\beta_k M_\mu' + \frac{\alpha_k - \beta_k\varepsilon_k'}{2} + \frac{1}{2(\alpha_k - \beta_k\varepsilon_k')}\\
&\qquad \times\Bigg[\alpha_k b_k + (1 + b_k)\Bigg(\sum_{i=n+k}^{2n-1}\sum_{j=0}^{i-n-k}\\
&\qquad\quad \times\binom{i-n-k}{j}\|P_i^{(j)}\|_c + \sum_{i=0}^{n+k-1}\|P_i\|_c\Bigg)\Bigg]^2\Bigg\}\|y^{(\mu)}\|^2.
\end{aligned}$$

Hence, the condition (a_n) holds, so does the $(a_n)'$ because of the similar method. Take the set $C_k(K)$ as in the Main Theorem. We have

COROLLARY 2. (i) *Suppose* $k \in \{0, 1, \ldots, n\}$ *and* $\lambda \in C_k(K)$. *Then*

(a) *The operator* $B_\lambda^{(k)}\colon (K, \|\cdot\|_{n+k}) \to (B_\lambda^{(k)}K, \|\cdot\|_{-(n+k)})$ *and its minimal extension*

$$\tilde{B}_\lambda^{(k)}\colon \left(\overline{K}^{\|\cdot\|_{n+k}}, \|\cdot\|_{n+k}\right) \to \left(\overline{B_\lambda^{(k)}K}^{\|\cdot\|_{-(n+k)}}, \|\cdot\|_{-(n+k)}\right)$$

are homeomorphisms.

(b) *When* $\beta_k = 0$, *both operators in* (a) *are linear homeomorphisms and*

$$\overline{B_\lambda^{(k)}K}^{\|\cdot\|_{-(n+k)}} = \left(\overline{K}^{\|\cdot\|_{n+k}}\right)^* \qquad (*)''$$

(c) *If* $C^{2n}[0,1]$ *is a real space and* $A_k^{(2)}$ *is a real operator, then for each* $\lambda \in C_k(K) \cap R$, *both operators in* (a) *are real homeomorphisms and* $(*)''$ *holds too.*

(ii) *Assume* $k = n$ *and* $\lambda \in C_n(K)$.

(a) *If* $A_n^{(2)}\colon (C^{2n}[0,1], \|\cdot\|_{2n}) \to (C[0,1], \|\cdot\|)$ *is a Lipschitz operator, then the operator* $B_\lambda^{(n)}\colon (K, \|\cdot\|_{2n}) \to (B_\lambda^{(n)}K, \|\cdot\|)$ *and its minimal continuous extension* $\overline{B_\lambda}^{(n)}\colon (\overline{K}^{\|\cdot\|_{2n}}, \|\cdot\|_{2n}) \to (\overline{B_\lambda^{(n)}K}^{\|\cdot\|}, \|\cdot\|)$ *are both homeomorphisms, and they are linear homeomorphisms in the case* $\beta_n = 0$

(b) *When* $\beta_n = 0$, *the operator*

$$B_\lambda^{(n)}\colon \left(K, \|\cdot\|_{C^{2n}}\right) \to \left(B_\lambda^{(n)}K, \|\cdot\|_C\right)$$

and its minimal continuous extension

$$\hat{B}_\lambda^{(n)}\colon \left(\overline{K}^{\|\cdot\|_{C^{2n}}}, \|\cdot\|_{C^{2n}}\right) \to \left(\overline{B_\lambda^{(n)}K}^{\|\cdot\|_C}, \|\cdot\|_C\right)$$

are linear homeomorphisms.

The result follows immediately from Lemma 5.2 *and the Main Theorem.*

Note 1. The result of [12] is the special case of Corollary 2(i)(a) $(k = n)$ with $N_\lambda \equiv \lambda$.

Note 2. By the Main Theorem $(n = 2)$ and Corollary 1, we know that there exist four or five regularity classes for the operators $A_\lambda + B$ or A_λ

in Section 1. Thus, based on the characteristics of the operator and subspace, we can select one or more regularity classes to control the flying vehicles in advance and then make it move smoothly.

Note 3. For the operator $B_\lambda = A + N_\lambda I$ with $A = d^{2n}/dx^{2n} + \sum_{i=0}^{2n-1} P_i(x)d^i/dx^i + B$, clearly, its linear part is in general form, so in case there is no perturbation $(B = 0)$, or little perturbation (i.e., $\varepsilon'_k \ll \alpha_k$ in (5.3)), or the differential order of B is less than $(2n - 1)$, for the moving stability in both directions of any long thin objects whose movement can be described by some homogeneous boundary problem of $2n$th-order differential equation with parameters, Corollary 2 is instructable.

Note 4. We can generalize the main result as follows

(a) If we change [0, 1] into $[a, b]$, then we have the similar result for the same operators and the subspaces of $C^{2n}[a, b]$ under similar conditions. In particular, if we take $K \subset C^{2n}[a, b]$ as $K = \{y(x) \in C^{2n}[a, b] \mid y^{(i)}_{x=a,b} = 0, i = 0, \dots, n-1\}$ *and* $k \in \{0, \dots, n\}$, $N(x) \in C^k[a, b]$, then for the operator B_λ, there are at least the first $(k + 1)$ regularity classes of (1.4) in Section 1. In the case $k = n$, there exist all regularity classes described in the Main Theorem and Corollary 2.

(b) The reason we take K as $K_{(i,j)}$ or $K_{(i_1,\dots,i_n)}$ in Corollaries 1 and 2 is that we usually should use the integration by parts that changes inner products into pure inner products besides some necessary homogeneous boundary condition. Once the boundary condition does not permit us to integrate by parts, we can solve the problem approximately as follows. For some small number $\varepsilon > 0$, by extending the subspace $K \subset C^{2n}[0, 1]$ into the subset $\tilde{K}$ of $C_c^{2n}(-\varepsilon, 1 + \varepsilon)$ with the truncation techniques of equations, then from (a), we know that the operator B_λ: $\tilde{K} \to B_\lambda \tilde{K}$ has the same regularity theorem as the Main Theorem. Restricted on K, we can get the approximate result for the operator B_λ: $K \to B_\lambda K$. Therefore, for long thin flying objects which can be described by the boundary value problem of $2n$th-order differential equation with parameters, it is practical to describe their stability in both directions in their moving process.

Note 5. For the subspaces K of $C^4[0, 1]$ or $C^{2n}[0, 1]$ in Corollaries 1 and 2, respectively, conditions (b_k) and (d_n) can be verified easily by applying Schwarz's inequality. If B_λ is linear, condition (e_n) always holds, and the constants in these conditions are irrelevant to the regularity point set $C_k(K)$. So in case that condition (R_k) holds, or $N(x) \in C^k[0, 1]$, we only need to verify the conditions (a_k) and $(a_k)'$. To do so, it is sufficient to use Schwarz's and Young's inequalities after integrating by parts. We explain this in the following two examples.

Let $k = n$, $N(x) \in C^n[0,1]$, $K_1 = K_{(0,\dots,n-1)}$, $K_2 = K_{(n,\dots,2n-1)}$,

$$A_n^{(1)} y = y^{(2n)} + \sum_{i=1}^{2n-1} P_i(x) y^{(2n-i)}, \qquad y \in C^{2n}[0,1],$$

$$A_n^{(2)} y = a_1 \sin\left(y^{(2n)}\right) + b_1 \frac{\left(y^{(2n)}\right)^2}{1 + \left(y^{(2n)}\right)^2}$$

$$+ \sin\left(\sum_{i=1}^{2n-1} R_i(x) y^{(i)}\right) + \sum_{i=1}^{2n-1} Q_i(x) \frac{y^{(i)}}{1 + \left(y^{(i)}\right)^2},$$

where the constants a_1 and b_1 satisfy $|a_1| + |b_1| < 1$, and P_i, Q_i, R_i all belong to $C[0,1]$ (real function). Define $B_\lambda = A_n^{(1)} + \beta_n A_n^{(2)} + N_\lambda I$ and $\beta_n = 0$ or 1.

EXAMPLE 1. If $\beta_n = 0$, $K = K_1$ or K_2, take

$$\varepsilon_0 = \frac{1}{2}, \qquad M_0 = 1 + \frac{\tilde{M}}{2}, \qquad M_i = \left(\tilde{M} + \frac{3}{2}\right) \|P_{2n-i}\|_c$$

$$(1 \le i \le 2n-1),$$

where

$$\tilde{M} = \sum_{i=1}^{2n-1} \|P_i\|_c.$$

Then the result in the Main Theorem for $k = n$ holds for the operator $B_\lambda = A_n^{(1)} + N_\lambda I$.

EXAMPLE 2. If $\beta_n = 1$, $K = K_1$ or K_2, take

$$\varepsilon_0 = \frac{1}{2}\left(1 - |a_1| - |b_1|\right),$$

$$M_0 = \frac{\left(1 + |a_1| + |b_1|\right)^2}{1 - |a_1| - |b_1|} + \frac{\overline{M}}{2}, \qquad \overline{M} = \sum_{i=1}^{2n-1} \left(\|P_i\|_c + 2\|Q_i\|_c + \|R_i\|_c\right),$$

$$M_i = \left(\frac{\overline{M} + 1}{1 - |a_1| - |b_1|} + \frac{1}{2}\right)\left(\|P_{2n-i}\|_c + 2\|Q_i\|_c + \|R_i\|_c\right)$$

$$(1 \le i \le 2n-1).$$

Then the result in the Main Theorem for $k = n$ holds for the operator $B_\lambda = A_n^{(1)} + A_n^{(2)} + N_\lambda I$.

Proof. We only verify the conditions (a_k) and $(a_k)'$ in Example 2. Let $A = A_n^{(1)} + A_n^{(2)}$. Then

$$\begin{aligned}
&\big(Ay - Az, y^{(2n)} - z^{(2n)} + (-1)^n (y - z)\big) \\
&\quad \geq \frac{3}{4}\big(1 - |a_1| - |b_1|\big)\|y^{(2n)} - z^{(2n)}\|^2 - \frac{\big(1 + |a_1| + |b_1|\big)^2}{1 - |a_1| - |b_1|}\|y - z\|^2 \\
&\qquad - \sum_{i=1}^{2n-1} \big(\|P_{2n-i}\|_c + 2\|Q_i\|_c + \|R_i\|_c\big) \\
&\qquad \times \|y^{(i)} - z^{(i)}\|\big(\|y^{(2n)} - z^{(2n)}\| + \|y - z\|\big). \qquad (5.5)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|y^{(i)} - z^{(i)}\|\big(\|y^{(2n)} - z^{(2n)}\| + \|y - z\|\big) \\
&\quad \leq \frac{1 - |a_1| - |b_1|}{4(\overline{M} + 1)}\|y^{(2n)} - z^{(2n)}\|^2 \\
&\qquad + \left(\frac{\overline{M} + 1}{1 - |a_1| - |b_1|} + \frac{1}{2}\right)\|y^{(i)} - z^{(i)}\|^2 + \frac{1}{2}\|y - z\|^2. \qquad (5.6)
\end{aligned}$$

Then from (5.5) and (5.6) we have for any $y, z \in C^{2n}[0, 1]$,

$$\begin{aligned}
&\big(Ay - Az, y^{(2n)} - z^{(2n)} + (-1)^n (y - z)\big) \\
&\quad \geq \frac{1}{2}\big(1 - |a_1| - |b_1|\big)\|y^{(2n)} - z^{(2n)}\|^2 \\
&\qquad - \left(\frac{\big(1 + |a_1| + |b_1|\big)^2}{1 - |a_1| - |b_1|} + \frac{\overline{M}}{2}\right)\|y - z\|^2 \\
&\qquad - \sum_{i=1}^{2n-1} \left(\frac{\overline{M} + 1}{1 - |a_1| - |b_1|} + \frac{1}{2}\right) \\
&\qquad \times \big(\|P_{2n-i}\|_c + 2\|Q_i\|_c + \|R_i\|_c\big)\|y^{(i)} - z^{(i)}\|^2.
\end{aligned}$$

Hence, the condition $(a_n)'$ holds. So does the condition (a_n) because $A0 = 0$.

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